

Plancherel-Rotach Asymptotics for q -Orthogonal Polynomials

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Abstract

We establish the Plancherel-Rotach-type asymptotics around the largest zero (the soft edge asymptotics) for some classes of polynomials satisfying three-term recurrence relations with exponentially increasing coefficients. As special cases, our results include this type of asymptotics for q^{-1} -Hermite polynomials of Askey, Ismail and Masson, q -Laguerre polynomials, and the Stieltjes-Wigert polynomials. We also introduce a one parameter family of solutions to the q -difference equation of the Ramanujan function.

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1 Introduction

Using the explicit representations and q -identities (in particular, the q -binomial theorem), Ismail [7] derived the complete asymptotics expansion of q^{-1} -Hermite, q -Laguerre, and Stieltjes-Wigert polynomials near their respective largest zeros. One of the main features is that the Ramanujan function $A_q(z)$, called the q -Airy function in [7], appears very naturally in all the results. These asymptotics are called the soft edge asymptotics in random matrices. There are two other types of Plancherel-Rotach asymptotics, the bulk scaling asymptotics where you normalize by the largest zero but keep x in the oscillatory region, and the tail asymptotics which is asymptotics beyond the largest and smallest zeros [18]. Ismail and Zhang [11]–[12] derived these asymptotics and found

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that the sine and cosine functions which appear in the bulk scaling asymptotics of the Hermite and Laguerre polynomial asymptotics [18] are now replaced by theta functions.

In this work, we will start with the more general three-term recurrence relations that include these classical q -orthogonal polynomials as special cases and derive the Plancherel-Rotach type asymptotics. A general characteristics of these recurrence relations is that the recurrence coefficients are growing exponentially with n , the degree of the polynomials. The orthogonality measures of these polynomials are not unique. Through careful scaling and transformation, these recurrence relations can be transferred into more trackable ones on which we base our study of the convergence. It must be emphasized that we use the recurrence relation directly in contrast to the previous works, [7], [11]–[12], which use more detailed information, such as explicit formulas.

The approach of deriving the asymptotic properties (e.g., ratio asymptotics, zero distributions, etc.) of orthogonal polynomials from assumptions on the asymptotic behavior of recurrence coefficients can be traced back to the work of Poincaré [16] and Blumenthal [3]. See the monograph in the Memoirs of AMS by Nevai [14] for a systematic treatment of the class $M(a, b)$ (the Nevai-Blumenthal class as it is referred to now, see, e.g., [13]) where the recurrence coefficients are assumed to be convergent. Starting with the assumptions on the recurrence coefficients, Van Assche [20] and Van Assche and Geronimo [22] obtained the Plancherel-Rotach asymptotics outside the oscillatory region (see also [21] and the references therein). Recently, Tulyakov [19] proposed a new method for obtaining the Plancherel-Rotach type asymptotics when the recurrence coefficients are rational functions in n (indeed, he dealt with more general difference equations of order p). This method allowed him to obtain the global picture of the asymptotic behavior of the solutions, and as a demonstration, he applied the method to the Hermite and Meixner polynomials.

In recent years the Riemann-Hilbert problem approach has been a powerful technique in determining asymptotics of orthogonal polynomials, see [4] for the description of the Riemann-Hilbert approach and [5] for the application to derive asymptotics of polynomials orthogonal with respect to exponential weights. So far the Riemann-Hilbert problem has not been successfully applied to q -orthogonal polynomials except in the case of the Stieltjes-Wigert polynomials when $q = e^{-1/(4n^2)}$ and n is the degree of the polynomial, see [2].

Throughout this work, we will assume that $0 < q < 1$.

Our model case is the q^{-1} -Hermite polynomials. Recall that the q^{-1} -Hermite polynomials $\{h_n(x | q)\}$ of Askey [1] and Ismail and Masson [10] satisfy the recurrence relation

$$(1.1) \quad 2xh_n(x | q) = h_{n+1}(x | q) + q^{-n}(1 - q^n)h_{n-1}(x | q),$$

with $h_0(x | q) = 1$ and $h_1(x | q) = 2x$. For detailed properties of the q^{-1} -Hermite polynomials see [8, Chapter 21]. Using a result in [9] it is easy to see that the largest zero of $h_n(x | q)$ is less than (but close to) $q^{-n/2}$. Let

$$(1.2) \quad x_n(t) = \frac{1}{2} [q^{-n/2}t - q^{n/2}/t].$$

Clearly, $x_{n\pm 1}(q^{\pm 1/2}t) = x_n(t)$. Set $x = x_n(t)$ in (1.1) to get

$$\left[q^{-n/2}t - \frac{q^{n/2}}{t} \right] h_n(x_n(t) \mid q) = h_{n+1}(x_{n+1}(q^{1/2}t) \mid q) + q^{-n}(1 - q^n)h_{n-1}(x_{n-1}(q^{-1/2}t) \mid q).$$

With $p_n(t) = t^{-n}q^{n^2/2}h_n(x_n(t) \mid q)$, the above recurrence becomes

$$(1.3) \quad \left(1 - \frac{q^n}{t^2} \right) p_n(t) = p_{n+1}(q^{1/2}t) + \frac{1 - q^n}{t^2} p_{n-1}(q^{-1/2}t).$$

If $p_n(t) \rightarrow f(t)$ uniformly on compact subsets of $\mathbb{C} \setminus \{0\}$, as $n \rightarrow \infty$, then $f(t)$ will be analytic in $\mathbb{C} \setminus \{0\}$ and satisfy

$$(1.4) \quad f(t) = f(q^{1/2}t) + \frac{1}{t^2}f(q^{-1/2}t)$$

Since $p_n(t)$ is a polynomial in $1/t^2$, f will have the Laurent expansion:

$$(1.5) \quad f(t) = \sum_{n=0}^{\infty} f_n t^{-2n}, \quad \text{with } f_0 = 1.$$

The substitution of f from (1.5) in (1.4) then equating coefficients of various powers of t implies that

$$f(t) = A_q(t^{-2}),$$

where $A_q(z)$ is the Ramanujan function [17] which plays the role of Airy function, as pointed out in [7]. The q -Airy function A_q satisfies q -difference equation:

$$(1.6) \quad A_q(z) = A_q(qz) - qzA_q(q^2z), \quad A_q(0) = 1$$

with power series expansion

$$(1.7) \quad A_q(z) = 1 + \sum_{k=1}^{\infty} \frac{q^{k^2}(-z)^k}{(1-q)(1-q^2) \cdots (1-q^k)}.$$

The purpose of this paper is to justify the above procedure for a more general class of orthogonal polynomials. More precisely we consider the family of orthogonal polynomials $\{p_n(x, c)\}$ defined by the recurrence relations

$$(1.8) \quad 2xp_n(x, c) = p_{n+1}(x, c) + q^{-nc}\beta_n(q, c)p_{n-1}(x, c),$$

where $c > 0$ and $\beta_n(q, c) = 1 + o(1)$ as $n \rightarrow \infty$. We will also remark on the case when $\beta_n(q, c) = 1 + \sum_{j=1}^{\infty} a_j(c)q^{\lambda_j n}$ for a sequence $\{\lambda_j\}$ with $\lambda_{j+1} > \lambda_j > 0, j = 1, 2, 3, \dots$, in Section 5. This generalizes the q^{-1} -Hermite model. A second result generalizes the Stieltjes-Wigert and q -Laguerre models. We consider the three term recurrence relation

$$(1.9) \quad xq^{2n+\alpha+1}p_n(x) = a_np_{n+1}(x) + b_np_{n-1}(x) + c_np_n(x),$$

where, modulo linear scaling of x by $ax + b$,

$$(1.10) \quad a_n = 1 + \sum_{k=1}^{\infty} a_{n,k} q^{\alpha_k n}, \quad b_n = 1 + \sum_{k=1}^{\infty} b_{n,k} q^{\beta_k n}, \quad c_n = 1 + \sum_{k=1}^{\infty} c_{n,k} q^{\gamma_k n},$$

where $\alpha_k > 0$, $\beta_k > 0$, and $\gamma_k > 0$ for all $k > 0$. We will see that this recurrence includes both the q -Laguerre and Stieltjes-Wigert polynomials as special cases in next section where we will also introduce a one parameter family of solutions to the q -difference equation in (1.6).

2 Main Results

As in the case of (1.1) we transform the polynomials $p_n(x, c)$ to the functions $s_n(t)$ defined as $s_n(t) = q^{n^2/2} t^{-n} p_n(x_n(t), c)$ with $x_n(t)$ given by (1.2). Thus $s_{-1}(t) = 0$, $s_0(t) = 1$, and

$$(2.1) \quad \left(1 - \frac{q^n}{t^2}\right) s_n(t) = s_{n+1}(q^{1/2}t) + \frac{1}{t^2} c(q, n) s_{n-1}(q^{-1/2}t), \quad n \geq 0,$$

where $|c(q, n)| \leq K$ and, for each q , $\lim_{n \rightarrow \infty} c(q, n) = 1$.

Note that $\{s_n(t)\}$ are polynomials in $1/t^2$ of degree $\lfloor n/2 \rfloor$.

Our main result is the proof of the convergence of $\{s_n(t)\}$. We remark that the recurrence (2.1) indeed includes the recurrence relation of the q^{-1} -Hermite (1.3) as a special case.

Theorem 2.1. *Let $\{s_n(t)\}$ be defined by the recurrence given by (2.1) with $s_{-1}(t) = 0$ and $s_0(t) = 1$. The limit $\lim_{n \rightarrow \infty} s_n(t)$ exists and the convergence is locally uniform for $t \in \overline{\mathbb{C}} \setminus \{0\}$.*

By the discussion immediately following (1.3), we have the following corollary.

Corollary 2.2. *Let $\{s_n(t)\}$ be defined by the recurrence given by (2.1). We have*

$$\lim_{n \rightarrow \infty} s_n(t) = A_q\left(\frac{1}{t^2}\right)$$

locally uniform for $t \in \overline{\mathbb{C}} \setminus \{0\}$.

The main ideas used in the proof of Theorem 2.1 for the treatment of the family that generalizes the q^{-1} -Hermite polynomials can be adapted to the family of recurrence relations (i.e., (1.9)) which include q -Laguerre and Stieltjes-Wigert polynomials as special cases. Due to the nature of non-symmetry, the implementation of these ideas become lengthier in the new situation. To transform (1.9), set

$$x = x_n(t) = q^{-2n-\alpha} t$$

and define $S_n(t) = q^{n^2}(-t)^{-n}p_n(x_n(t))$. Then the recurrence relation (1.9) can be written as

$$(2.2) \quad S_n(t) = (1 - q^{n+1})S_{n+1}(q^2t) + (1 + q)c(q, n)(qt)^{-1}S_n(t) + qd(q, n)t^{-2}S_{n-1}(q^{-2}t),$$

where $|c(q, n)| \leq K$ and $|d(q, n)| \leq K^2$ with $\lim_{n \rightarrow \infty} c(q, n) = \lim_{n \rightarrow \infty} d(q, n) = 1$. It is clear that $S_n(t)$ is a polynomial in $1/t$ with

$$(2.3) \quad S_n(\infty) = 1/(q; q)_n.$$

To see that q -Laguerre polynomials $\{L_n^{(\alpha)}(x; q)\}$ and Stieltjes-Wigert polynomials $\{S_n(x; q)\}$ are special cases of (1.9) and (1.10), we recall that $\{L_n^{(\alpha)}(x; q)\}$ is generated by

$$(2.4) \quad \begin{aligned} -xq^{2n+\alpha+1}L_n^{(\alpha)}(x; q) &= (1 - q^{n+1})L_{n+1}^{(\alpha)}(x; q) + q(1 - q^{n+\alpha})L_{n-1}^{(\alpha)}(x; q) \\ &\quad - [(1 - q^{n+1}) + q(1 - q^{n+\alpha})]L_n^{(\alpha)}(x; q) \end{aligned}$$

with initial conditions

$$L_0^{(\alpha)}(x; q) = 1, \quad L_1^{(\alpha)}(x; q) = \frac{1 - q^{\alpha+1} - xq^{\alpha+1}}{1 - q};$$

while the Stieltjes-Wigert polynomials $\{S_n(x; q)\}$ are defined by

$$(2.5) \quad -xq^{2n+1}S_n(x; q) = (1 - q^{n+1})S_{n+1}(x; q) - [1 + q - q^{n+1}]S_n(x; q) + qS_{n-1}(x; q),$$

with the initial conditions

$$S_0(x; q) = 1, \quad S_1(x; q) = \frac{1 - qx}{1 - q}.$$

Recall that $(a; q)_\infty = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) \cdots = \prod_{n=0}^\infty (1 - aq^n)$.

Theorem 2.3. *Let $\{S_n(t)\}$ be defined by (2.2) with initial conditions $S_0(t) = 1$ and $S_1(t) = [1 - q(1 + q)c(q, 1)]/t/(1 - q)$. Then the limit $\lim_{n \rightarrow \infty} S_n(t)$ exists and*

$$\lim_{n \rightarrow \infty} S_n(t) = \frac{1}{(q; q)_\infty} A_q\left(\frac{1}{t}\right),$$

locally uniform for $t \in \overline{\mathbb{C}} \setminus \{0\}$.

We have seen that $y = A_q(z)$ satisfies the functional equation

$$(2.6) \quad y(z) - y(qz) + qzy(q^2z) = 0.$$

It is worth noting that the family of functions

$$(2.7) \quad F_q(z; s) := e^{i\pi s} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(n+s)^2}}{(q; q)_{n+s}} z^{n+s}$$

are solutions to (2.6) for all s . The notation used in the above equation is (cf. [6, 8])

$$(2.8) \quad (a; q)_b = \frac{(a; q)_\infty}{(aq^b; q)_\infty}.$$

Observe that $F_q(z; s)$ is periodic in s of period 1, so there is no loss of generality in taking s in the strip $\text{Re}(s) \in [0, 1)$. Indeed, $A_q(z) = F_q(z; 0)$.

3 Proof of Theorem 2.1

Let $\{s_n(t)\}$ be given by the recurrence relation (2.1) with $s_{-1}(t) = 0$ and $s_0(t) = 1$. We will need the following lemma for the proof of Theorem 2.1.

Lemma 3.1. *If there exist constants $\rho > 0$ and $M > 0$ such that*

$$(3.1) \quad \max_{|t|=\rho} |s_n(t)| \leq M$$

for all $n \geq 0$, then $\{s_n(t)\}$ is a normal family on $\overline{\mathbb{C}} \setminus \{0\}$.

Proof. Since $s_n(t)$ are polynomials in $1/t^2$, they are all analytic on $\overline{\mathbb{C}} \setminus \{0\}$. So, by the maximal modulus principle for analytic functions, (3.1) implies that

$$(3.2) \quad \max_{|t| \geq \rho} |s_n(t)| \leq M$$

for all $n \geq 0$. Next, we show that (3.2) implies that

$$(3.3) \quad \max_{|t| \geq q^{1/2}\rho} |s_n(t)| \leq \left(1 + \frac{1+K}{\rho^2}\right)M$$

for all $n \geq 2$.

Indeed, from the recurrence relation (2.1),

$$s_{n+1}(q^{1/2}t) = \left(1 - \frac{q^n}{t^2}\right) s_n(t) - \frac{1}{t^2} c(q, n) s_{n-1}(q^{-1/2}t).$$

Thus, using (3.2), for $|t| = \rho$, we have

$$|s_{n+1}(q^{1/2}t)| \leq \left(1 + \frac{1}{|t|^2}\right) M + \frac{K}{|t|^2} M = \left(1 + \frac{1+K}{|t|^2}\right) M,$$

for $n \geq 1$. This proves (3.3). Repeating the argument above, we obtain that, for $k = 1, 2, \dots$, we have

$$(3.4) \quad \max_{|t| \geq q^{k/2}\rho} |s_n(t)| \leq M \prod_{j=1}^k \left(1 + \frac{1+K}{\rho^2 q^{j-1}}\right)$$

for $n \geq k+1$.

Since $\lim_{k \rightarrow \infty} q^{k/2}\rho = 0$, we see that the sequence $\{s_n(t)\}$ is uniformly bounded on compact subsets of $\overline{\mathbb{C}} \setminus \{0\}$. \square

We need one more auxiliary result for the proof of Theorem 2.1.

Lemma 3.2. *We have, for $|t| \geq 1$,*

$$(3.5) \quad |s_n(t)| \leq \prod_{k=0}^n (1 + q^k) A_q\left(-\frac{K}{|t|^2}\right)$$

for all $n \geq 0$.

Proof. We use induction. First it is trivial to verify that (3.5) holds for $n = 0$. Now, assume (3.5) is true for n and let us verify that (3.5) is also true when n is replaced by $n + 1$.

From (2.1), we have

$$s_{n+1}(t) = (1 - \frac{q^{n+1}}{t^2})s_n(q^{-1/2}t) - \frac{q}{t^2}c(n, q)s_{n-1}(q^{-1}t).$$

Thus, for $|t| \geq 1$,

$$\begin{aligned} |s_{n+1}(t)| &\leq (1 + q^{n+1})|s_n(q^{-1/2}t)| + \frac{Kq}{|t|^2}|s_{n-1}(q^{-1}t)| \\ &\leq (1 + q^{n+1}) \prod_{k=0}^n (1 + q^k) A_q(-\frac{Kq}{|t|^2}) + \frac{Kq}{|t|^2} \prod_{k=0}^{n-1} (1 + q^k) A_q(-\frac{Kq^2}{|t|^2}) \\ &\leq \prod_{k=0}^{n+1} (1 + q^k) \left(A_q(-\frac{Kq}{|t|^2}) + \frac{Kq}{|t|^2} A_q(-\frac{Kq^2}{|t|^2}) \right) \\ &= \prod_{k=0}^{n+1} (1 + q^k) A_q(-\frac{K}{|t|^2}), \end{aligned}$$

where in the last equality, we have applied (1.6) with $z = -K/|t|^2$. This completes the proof of the proposition. \square

Now, we are ready for the proof of Theorem 2.1.

Proof of Theorem 2.1. First, note that (i) the infinite product in (3.5) converges as $n \rightarrow \infty$, (ii) $A_q(-\frac{K}{|t|^2}) > 0$, and (iii) when $|t| = 1$,

$$A_q(-\frac{K}{|t|^2}) \leq A$$

for some $A > 0$. Hence, with the help of Lemmas 3.1 and 3.2, we see that $\{s_n(t)\}$ is a normal family for $t \in \overline{\mathbb{C}} \setminus \{0\}$.

To establish the convergence of $\{s_n(t)\}$, we study the coefficients of $s_n(t)$. Recall that $s_n(t)$ is a polynomial of degree $\lfloor n/2 \rfloor$ in $1/t^2$. Write

$$(3.6) \quad s_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{n,k} \frac{1}{t^{2k}}.$$

Then, the fact that $\{s_n(t)\}$ is a normal family implies that $\{a_{n,k}\}$ is a bounded set. Assume that

$$(3.7) \quad |a_{n,k}| \leq M, \quad \text{for } 0 \leq k \leq \lfloor n/2 \rfloor, \quad n = 0, 1, 2, \dots$$

Next, we show that, for every fixed k ,

$$(3.8) \quad \lim_{n \rightarrow \infty} a_{n,k} =: f_k$$

exists.

For $n \geq 0$, using (3.6) in (2.1), we get

$$(1 - \frac{q^n}{t^2}) \sum_{k=0}^{\lfloor n/2 \rfloor} a_{n,k} \frac{1}{t^{2k}} = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} a_{n+1,k} \frac{1}{t^{2k} q^k} + \frac{1}{t^2} c(q, n) \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n-1,k} \frac{q^k}{t^{2k}}$$

Comparing the constant terms on both sides of the above equation, we obtain

$$a_{n,0} = a_{n+1,0}, \quad n = 0, 1, 2, \dots$$

From $s_0(t) = 1$, we get $a_{0,0} = 1$, so, we have

$$(3.9) \quad a_{n,0} = 1, \quad n = 0, 1, 2, \dots$$

Next, for $k > 0$, comparing the coefficients of $1/t^{2k}$, we get, for large n ,

$$a_{n,k} - q^n a_{n,k-1} = \frac{a_{n+1,k}}{q^k} + q^{k-1} c(q, n) a_{n-1,k-1}$$

Or, equivalently, for large n ,

$$(3.10) \quad q^k a_{n,k} - q^{n+k} a_{n,k-1} = a_{n+1,k} + q^{2k-1} c(q, n) a_{n-1,k-1}.$$

Replacing n by $n+l$ in (3.10), we get

$$(3.11) \quad q^k a_{n+l,k} - q^{n+l+k} a_{n+l,k-1} = a_{n+l+1,k} + q^{2k-1} c(q, n+l) a_{n+l-1,k-1}.$$

Subtracting (3.11) from (3.10) and re-arranging the terms, we get, for large n ,

$$\begin{aligned} a_{n+1,k} - a_{n+l+1,k} &= q^k (a_{n,k} - a_{n+l,k}) - q^{n+k} (a_{n,k-1} - q^l a_{n+l,k-1}) \\ &\quad - q^{2k-1} [c(q, n) a_{n-1,k-1} - c(q, n+l) a_{n+l-1,k-1}]. \end{aligned}$$

This, together with (3.7) and the fact that $0 < q < 1$, gives us

$$(3.12) \quad \begin{aligned} |a_{n+1,k} - a_{n+l+1,k}| &\leq q^k |a_{n,k} - a_{n+l,k}| + q^{n+k} 2M + q^{2k-1} |a_{n-1,k-1} - a_{n+l-1,k-1}| \\ &\quad + q^{2k-1} M (|c(q, n) - 1| + |c(q, n+l) - 1|). \end{aligned}$$

With the help of (3.12), we are ready to prove (3.8) by induction on k .

When $k = 0$, equations in (3.9) give us

$$(3.13) \quad \lim_{n \rightarrow \infty} a_{n,0} = 1.$$

Now, assume (3.8) is true when k is replaced by $k-1$. Then, for any $\varepsilon > 0$, there exists a number $N > 0$ such that

$$(3.14) \quad |a_{n-1,k-1} - a_{n+l-1,k-1}| < \varepsilon, \quad n \geq N, \quad l \geq 0.$$

From $\lim_{n \rightarrow \infty} c(q, n) = 1$, we may choose N large enough to ensure that

$$(3.15) \quad |c(q, n) - 1| < \varepsilon, \quad n \geq N.$$

Thus, for $n \geq N$, (3.12) implies

$$(3.16) \quad |a_{n+1,k} - a_{n+l+1,k}| \leq q^k |a_{n,k} - a_{n+l,k}| + q^{n+k} 2M + q^{2k-1} \varepsilon (1 + 2M).$$

After repeatedly using (3.16) m times, we arrive at

$$\begin{aligned} |a_{n+1,k} - a_{n+l+1,k}| &\leq q^{(m+1)k} |a_{n-m,k} - a_{n-m+l,k}| \\ &\quad + q^{n+k} 2M \sum_{j=0}^m q^{j(k-1)} + q^{2k-1} \varepsilon (1 + 2M) \sum_{j=0}^m q^{jk}, \\ &\leq q^{(m+1)k} 2M + q^{n+k} 2M \frac{1}{1 - q^{k-1}} + q^{2k-1} \varepsilon (1 + 2M) \frac{1}{1 - q^k}, \quad n - m \geq N, \quad l \geq 0. \end{aligned}$$

Take $m = \lfloor \sqrt{n} \rfloor$. Then $\lim_{n \rightarrow \infty} m = \infty$ and $\lim_{n \rightarrow \infty} (n - m) = \infty$. Therefore, for large n and for all $l \geq 0$,

$$\overline{\lim}_{n \rightarrow \infty} |a_{n,k} - a_{n+l,k}| \leq \varepsilon (1 + 2M) \frac{q^{2k-1}}{1 - q^k}.$$

So,

$$(3.17) \quad \lim_{n \rightarrow \infty} (a_{n,k} - a_{n+l,k}) = 0$$

uniformly in $l \geq 0$. Thus, $\{a_{n,k}\}_{n \geq k}$ is a Cauchy sequence. Hence, (3.8) is true for any $k = 0, 1, 2, \dots$

Finally, we need to derive the convergence of $\{s_n(z)\}$. Although this can be done by a bounded convergence argument based on (3.7) and (3.8), we choose the following more elementary argument for its simplicity: Note that every subsequence $\{s_n(t)\}_{n \in \Lambda}$ of $\{s_n(t)\}$ has a sub-subsequence $\{s_n(t)\}_{n \in \Lambda_1}$ ($\Lambda_1 \subseteq \Lambda$) that converges locally uniformly in $\overline{\mathbb{C}} \setminus \{0\}$. Assume

$$f(t) := \lim_{n \rightarrow \infty, n \in \Lambda_1} s_n(t).$$

Then

$$f^{(j)}(t) = \lim_{n \rightarrow \infty, n \in \Lambda_1} s_n^{(j)}(t).$$

Evaluating at $t = \infty$, we see that

$$f(t) = \sum_{j=0}^{\infty} f_j \frac{1}{t^{2j}}.$$

Therefore, the whole sequence $\{s_n(t)\}$ must converge to the same function $f(t)$ locally uniformly in $\overline{\mathbb{C}} \setminus \{0\}$. \square

4 Proof of Theorem 2.3

We need to recall some notations from q -theory and solve a q -difference equation.

4.1 The Functional Equation

Recall that (see (2.8))

$$(4.1) \quad (a; q)_0 = 1, \quad (a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j),$$

and we note the Euler identity [6, Page xvi, (18)]

$$(4.2) \quad \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_{\infty}}, \quad |z| < 1.$$

We will need to solve the functional equation

$$(4.3) \quad f(z) = f(zq^2) + \frac{aq}{z^2} f(zq^{-2}) + a \frac{1+q}{qz} f(z), \quad a = \pm 1.$$

Proposition 4.1. *Up to a scaling factor, the above functional equation has a solution:*

- (i) *when $a = 1$, $f(z) = A_q(\frac{1}{z})$, and*
- (ii) *when $a = -1$, $f(1/z)$ is an entire function with positive Taylor series coefficients.*

Proof. Assume that $f(z) = \sum_{n=0}^{\infty} f_n z^{-n}$. Since $f_0 \neq 0$ is a scaling factor there is no loss of generality in assuming $f_0 = 1$. By equating coefficients of z^{-n} on both sides of (4.3) we see that

$$(4.4) \quad a(1 - q^{-2n})f_n = q^{2n-1}f_{n-2} + (1+q)q^{-1}f_{n-1}, \quad n > 0,$$

with $f_1 = aq/(q-1)$. Now, in order to normalize the coefficients in difference equation (4.4) above, set $f_n = g_n q^{n^2}$. Thus $g_0 = 1, g_1 = a/(q-1)$ and

$$g_n(q^{2n} - 1) = aqg_{n-2} + a(1+q)g_{n-1}, \quad n > 0.$$

We now solve the above recursion using generating functions, so we set $G(z) = \sum_{n=0}^{\infty} g_n z^n$. In view of the initial conditions, the functional equation for $G(z)$ is

$$(4.5) \quad G(z) = G(q^2 z) / [1 + a(1+q)z + aqz^2].$$

If $a = 1$ then the denominator in (4.5) factors as $(1+z)(1+qz)$, and iterating (4.5) with the initial condition $G(0) = 1$ gives

$$G(z) = \frac{1}{(-z; q)_{\infty}} = \sum_{k=0}^{\infty} \frac{(-z)^k}{(q; q)_k},$$

by the Euler identity. In the last step we used (4.2). This leads to $f(z) = A_q(1/z)$.

On the other hand when $a = -1$, the denominator in (4.5) factors as

$$1 - (1 + q)z - qz^2 = (1 - z/\alpha)(1 - z/\beta), \quad \text{with } \alpha < 0 < \beta \text{ and } \beta < |\alpha|.$$

In this case

$$G(z) = \frac{1}{(z/\alpha; q^2)_\infty (z/\beta; q^2)_\infty}.$$

Since $0 < \beta < |\alpha|$, we see that the singularity of G with the smallest modulus is β . Applying Darboux's asymptotic method, [15], we conclude that

$$g_n = \frac{\beta^{-n}}{(q^2; q^2)_\infty (\beta/\alpha; q^2)_\infty} [1 + o(1)].$$

Thus $f_n = g_n q^{n^2}$ and $f(1/z)$ is an entire function of z . Moreover the recurrence relation (4.4) and the initial conditions show that $f_n > 0$. Therefore the function $f(1/z)$ is an entire function of z with positive Taylor series coefficients. \square

Proof of Theorem 2.3. We follow the ideas used in the proof of Theorem 2.1. We will skip similar argument. The main task is to show that the family $\{S_n(t)\}$ defined by (2.2) is a normal family for $t \in \mathbb{C} \setminus \{0\}$.

Let $f^b(t)$ be the solution with $f^b(\infty) = 1$ in Proposition 4.1 when $a = -1$. We claim that for $|t| \geq 1$,

$$(4.6) \quad |S_n(t)| \leq \frac{f^b(|t|/K)}{\prod_{k=1}^n (1 - q^k)},$$

for $n = 0, 1, 2, \dots$

Let us use induction to verify (4.6) for $n = 0, 1, 2, \dots$. Clearly, from the initial conditions and the fact that $f^b(|t|/K) = 1 + Kq/((1 - q)|t|) + (\text{positive terms})$, we see that (4.6) is true when $n = 0, 1$. Now assume that (4.6) is true for n .

From (2.2), we have

$$(1 - q^{n+1})S_{n+1}(t) = S_n(q^{-2}t) - (1 + q)c(q, n)(q^{-1}t)^{-1}S_n(q^{-2}t) - q^5 d(q, n)t^{-2}S_{n-1}(q^{-4}t).$$

So, for $|t| \geq 1$,

$$\begin{aligned} |(1 - q^{n+1})S_{n+1}(t)| &\leq |S_n(q^{-2}t)| + \frac{(1 + q)q}{|t|} |S_n(q^{-2}t)| + \frac{q^5}{|t|^2} |S_{n-1}(q^{-4}t)| \\ &\leq \frac{f^b(q^{-2}|t|/K)}{\prod_{k=1}^n (1 - q^k)} \left(1 + \frac{K(1 + q)q}{|t|}\right) + \frac{K^2 q^5}{|t|^2} \frac{f^b(q^{-4}|t|/K)}{\prod_{k=1}^{n-1} (1 - q^k)} \\ &\leq \frac{1}{\prod_{k=1}^n (1 - q^k)} \left[\left(1 + \frac{K(1 + q)q}{|t|}\right) f^b\left(\frac{q^{-2}|t|}{K}\right) + \frac{K^2 q^5}{|t|^2} f^b\left(\frac{q^{-4}|t|}{K}\right) \right] \\ &= \frac{1}{\prod_{k=1}^n (1 - q^k)} f^b\left(\frac{|t|}{K}\right), \end{aligned}$$

where in the last step, we used the functional equation ((4.3) with $a = -1$) satisfied by f^b with $z = q^{-2}|t|/K$ and the fact that $f^b(\frac{1}{z})$ has positive Taylor series coefficients. This implies (4.6) holds when n is replaced by $n + 1$. Therefore, by induction, (4.6) is true for all $n = 0, 1, 2, \dots$

The rest of the proof goes like the one for Theorem 2.1: From the claim, we can easily show that $\{S_n(t)\}$ is a normal family for $t \in \overline{\mathbb{C}} \setminus \{0\}$ and from here and working with the recurrence relations, we can show (by a similar argument as given in the proof of Theorem 2.1) that the coefficients of $S_n(t)$ converge: if

$$S_n(t) = \sum_{k=0}^n a_{n,k} t^{-k}$$

then

$$\lim_{n \rightarrow \infty} a_{n,k} = f_k, \quad \text{for some } f_k, \quad k = 0, 1, 2, \dots,$$

which immediately yields the convergence of the whole sequence $\{S_n(t)\}$.

Finally, writing the limit as $f(t)$: $f(t) := \lim_{n \rightarrow \infty} S_n(t)$, and let $n \rightarrow \infty$ on both sides of (2.2) to get

$$f(t) = f(q^2 t) + (1 + q)(qt)^{-1} f(t) + qt^{-2} f(q^{-2} t).$$

From this, with the fact that $\lim_{n \rightarrow \infty} S_n(\infty) = 1/(q; q)_\infty$, Proposition 4.1 when $a = 1$ implies that $f(t) = A_q(\frac{1}{t})/(q; q)_\infty$. \square

5 Error Terms

Unlike the asymptotics of the q^{-1} -Hermite polynomials [7], the terms beyond the main term in the asymptotic expansion of $s_n(t)$ seem to be intricate.

In (2.1) we assume that $c(q, n)$ has the convergent expansion

$$(5.1) \quad c(q, n) = \sum_{k=0}^m c_k q^{nk} + o(q^{nm}), \quad c_0 = 1.$$

We further assume that $s_n(t)$ has the asymptotic expansion

$$(5.2) \quad s_n(t) = \sum_{k=0}^m f_{n,k}(1/t^2) q^{nk} + o(q^{nm}).$$

Now substitute for s_n from (5.2) in (2.1) and equate the coefficients of q^{nk} . When $k = 0$ we see that $f_{n,0}(1/t^2)$ solves

$$(5.3) \quad y(u) - y(u/q) - uy(qu) = 0,$$

with $u = 1/t^2$. When $k > 0$ we conclude that

$$(5.4) \quad \begin{aligned} & f_{n,k}(u) - q^k f_{n+1,k}(u/q) - uq^{-k} f_{n-1,k}(qu) \\ &= u f_{n,k-1}(u) + u \sum_{j=0}^{k-1} c_{k-j} q^{-j} f_{n-1,j}(qu), \end{aligned}$$

for $k = 1, 2, \dots$. Since $s_n(t)$ is a polynomial in $1/t^2$ we expect $f_{n,k}(u)$ to be analytic in u in a neighborhood of $u = 0$. Thus $f_{n,0}(u) = A_q(u)$. Note that (5.4) implies $f_{n,k}(0) = 0$ for $k > 0$, since $s_n(\infty) = 1 = A_q(0)$. Let $g_{n,k}(u) = f_{n,k}(u)/u$, for $k > 0$. Thus (5.4) gives

$$(5.5) \quad g_{n,1}(u) - g_{n+1,1}(u/q) - u g_{n-1,1}(qu) = A_q(u) + c_1 A_q(qu).$$

The case $c(q, n) = 1 - q^n$ is very exceptional. In this case Ismail [7] showed that

$$(5.6) \quad s_n(t) = \sum_{k=0}^{\infty} \frac{q^{j(j+1)/2}}{(q; q)_j t^{2j}} A_q(q^j/t^2) q^{jn}.$$

In this case (5.4) becomes

$$(5.7) \quad \begin{aligned} f_{n,k}(u) - q^k f_{n+1,k}(u/q) - u q^{-k} f_{n-1,k}(qu) \\ = u f_{n,k-1}(u) - u q^{1-k} f_{n-1,k-1}(qu), \end{aligned}$$

and by induction $f_{n,k}(u)$ must have the form $u^k g_{n,k}(u)$ and $g_{n,k}$ satisfy

$$(5.8) \quad g_{n,1}(u) - g_{n+1,1}(u/q) - u g_{n-1,1}(qu) = g_{n,k-1}(u) - g_{n-1,k-1}(qu),$$

which leads to the solution given by (5.6).

The analysis of (5.5) in general seems to be complicated. As an illustration we first consider the case $c(q, n) = 1 + c_1 q^n$. We then let $g_{n,1} = \sum_{j=0}^{\infty} \zeta_{n,j} u^j$ and substitute it in (5.5). Thus $\zeta_{n+1,0} = \zeta_{n,0} + 1 + c_1$, and we conclude that $\zeta_{n,0} = n\zeta_{0,0} + n(1 + c_1)$. Therefore $\zeta_{n,0}$ is independent of n if and only if $c_1 = -1$. When $c_1 = -1$ we are led to the expansion (5.6).

For general c_1 we make the Ansatz $g_{n,1}(u) = nF(u) + G(u) + o(1)$ which leads to $F(u) = A_q(u)$ and that $G(u)$ solves

$$(5.9) \quad G(u) - G(u/q) - uG(qu) = 2A_q(u/q) + c_1 A_q(qu).$$

Writing $G(u) = \sum_{j=0}^{\infty} \lambda_j u^j$ we find the following two-term recurrence relation for λ_n

$$(5.10) \quad (-1)^j \lambda_j q^{j-j^2} = \frac{1}{1-q^j} (-1)^{j-1} q^{j-1-(j-1)^2} \lambda_{j-1} + 2(-1)^j \frac{1+c_1 q^{2j}}{1-q^j},$$

which is easy to solve. This process can be iterated but it gets complicated as we proceed to higher k .

The error analysis for (2.2) is even harder. Ismail [7] proved

$$(5.11) \quad S_n(q^{-2n}t; q) = \frac{1}{(q; q)_{\infty}} \sum_{s=0}^{\infty} \frac{(-1)^s}{(q; q)_s} q^{\binom{s+1}{2}} q^{ns} A_q(q^{-s}/t),$$

and

$$(5.12) \quad \begin{aligned} \frac{q^{n^2} L_n^{(\alpha)}(x_n(t); q)}{(-t)^n} &= \frac{1}{(q; q)_{\infty}} \sum_{m=0}^{\infty} \frac{q^{m/2}}{(q; q)_m} q^{mn} \\ &\times \sum_{s=0}^m \frac{(q; q)_m}{(q; q)_s (q; q)_{m-s}} (-1)^{m-s} q^{s\alpha + (m-2s)^2/2} A_q(q^{2s-m}/t). \end{aligned}$$

Since $S_n(\infty) = 1/(q; q)_n$ we let $S_n(t) = \frac{1}{(q; q)_n} \sum_{k=0}^{\infty} f_{n,k}(u) q^{nk}$ where $u = 1/t$. Equation (2.2) leads to

$$(5.13) \quad f_{n,0}(u) = f_{n+1,0}(u/q^2) + \frac{(1+q)}{q} u f_{n,0}(u) + q u^2 f_{n-1,0}(q^2 u).$$

Any constant times $A_q(u)$ will solve (5.13) as per (i) of Proposition 4.1. In general, we have

$$(5.14) \quad \begin{aligned} f_{n,k}(u) = & f_{n+1,k}(u/q^2) q^k + \frac{(1+q)u}{q} \sum_{j=0}^k c_{k-j} f_{n,j}(u) \\ & + q u^2 \sum_{j=0}^k d_{k-j} q^{-j} f_{n-1,j}(u q^2) - u^2 \sum_{j=0}^{k-1} d_{k-1-j} q^{-j} f_{n-1,j}(u q^2). \end{aligned}$$

We do not expect $f_{n,k}$ to depend on n . One reason is the following. In (2.2), let $S_n(t) = \frac{1}{(q; q)_n} + \sum_{j=1}^n a_{n,j} t^{-j}$. We then have

$$a_{n,1} = (1 - q^{n+1}) a_{n+1,1} q^{-2} + \frac{1+q}{q(q; q)_n} c(q, n).$$

Thus

$$q^{-2n} (q; q)_n a_{n,1} = q^{-2n-2} (q; q)_{n+1} a_{n+1,1} + \frac{1+q}{q} c(q, n) q^{-2n}$$

which gives $q^{-2n} (q; q)_n a_{n,1} = (1 + 1/q) \sum_{j=0}^{n-1} c(q, j) q^{-2j}$.

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